ON THE SIZE OF CERTAIN NUMBER-THEORETIC FUNCTIONS

BY

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1. **Summary of results.** Let p denote a prime number, m and n positive integers, and ω, ω_1, ω_2 real numbers. Let $f(m)$ be an additive number-theoretic function, so that $f(mn)=f(m)+f(n)$ if $(m, n)=1$. Suppose that $f(p^n)=f(p)$ and $|f(p)|\leqq 1$. Then clearly $f(m)=\sum_{p|m} f(p)$. Let

$$A_n = \sum_{p<n} f(p)/p, \qquad B_n = \left(\sum_{p<n} f^2(p)/p\right)^{1/2},$$

$$D(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\omega} e^{-x^2/2}dx,$$

and assume that $B_n \to \infty$ with n.

Erdös and Kac [7][2] have proved the following theorem: *The number of $m\leqq n$ for which $f(m)<A_n+\omega B_n$ is $nD(\omega)+o(n)$, as $n\to\infty$.* The present paper is concerned with the proofs of a number of related results. In §2 there is given a simpler proof of the special case of the above theorem in which $f(m)$ is taken to be $\nu(m)$, the number of distinct prime divisors of m. The simplification lies in that part of the proof using Brun's method; the central limit theorem from the theory of probability is still used. Moreover, the error term is improved, the term $o(n)$ being replaced by $O(n \log_3 n/(\log_2 n)^{1/4})$. (The symbol $\log_k n$ will be used throughout to denote the kth iterate of $\log n$.)

It is shown in §3 that a similar reduction of the error term can be effected in a theorem of Kac [11], which says that the number of $m\leqq n$ for which

$$d(m) < 2^{\log_2 n+\omega(\log_2 n)^{1/2}}$$

is $nD(\omega)+o(n)$. Here $d(m)$ is the number of divisors of m.

It is probable that the error is actually $O(n/(\log_2 n)^{1/2})$, but this result cannot be obtained without essential modification of the method used here.

§4 is devoted to a proof that $f(m)$, $f(m+1)$ are statistically independent, with Gaussian distribution. This was stated without proof by Erdös [6].

In §5 this theorem is applied in proving that the number of $m\leqq n$ for which $\nu(m)<\nu(m+1)+\omega(2\log_2 n)^{1/2}$ is $nD(\omega)+o(n)$. By the method of [11] it follows that the number of $m\leqq n$ for which

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[2] Numbers in brackets refer to the bibliography at the end of this paper.

$$d(m) < 2^{\omega(2 \log_2 n)^{1/2}} d(m + 1)$$

is also $nD(\omega)+o(n)$. These results are generalizations of theorems proved by Erdös [4]; these theorems are obtained by putting $\omega=0$:

The density of integers m for which $\nu(m)<\nu(m+1)$ is 1/2.

The density of integers m for which $d(m)<d(m+1)$ is 1/2.

The following remarks might serve to orient the reader not already familiar with theorems of the above type. Let $f(m)$ be additive, and denote by $N(c, n)$ the number of $m<n$ for which $f(m)<c$. The function $\psi(c)$ is called the distribution function of $f(m)$ if $\psi(-\infty)=0$, $\psi(\infty)=1$, and if for every finite c,

$$\psi(c) = \lim_{n\to\infty} N(c, n)/n.$$

Thus if $f(m)$ has a distribution function, $\psi(c)$ is, for fixed c, the density of integers for which $f(m)<c$. Clearly not every additive function has a distribution function, since $f(m)=\log m$ has none. Erdös and Wintner [8] showed that a necessary and sufficient condition for the existence of a distribution function is that both the inequalities

$$\sum_p \frac{f'(p)}{p} < \infty \quad \text{and} \quad \sum_p \frac{(f'(p))^2}{p} < \infty$$

hold, where $f'(p)=f(p)$ if $|f(p)| \leqq 1$ and $f'(p)=1$ otherwise. Hence if either or both of these inequalities fail to hold, we must adopt a different approach; instead of speaking of the density of integers with a certain property we estimate the number of integers which are less than n which satisfy a certain condition, and this condition itself depends upon n. This leads, for example, to the theorem of Erdös [5] that the number of $m\leqq n$ for which $\nu(m)<\log\log n$ is $n/2+o(n)$. The extension from this theorem to that of [7] corresponds, in the case of a function which has a distribution function, to an extension from a theorem giving its value for some particular c to one exhibiting the entire distribution function.

For use in §4 we now state a theorem of J. B. Rosser and W. J. Harrington [14] which exhibits a result obtained by the use of Brun's method. It may be put in the following form:

THEOREM A. *Hypotheses*:

(a) A, Q *are absolute constants.*

(b) $k, q_1, q_2, \cdots, q_T$ *are positive integers relatively prime in pairs*; α_i *are integers with* $0<\alpha_i<q_i$ *for* $0<i\leqq T$; a_{ij} *are integers,* $1\leqq i\leqq T$, $1\leqq j\leqq \alpha_i$, *such that for* $j\neq k$, $a_{ij}\not\equiv a_{ik} \pmod{q_i}$; *$f$ is a function having an integral value for integral argument*; $N_t(k)=\sum g(m)$, *where the summation is over all integers m such that simultaneously m satisfies some fixed condition independent of* $t, k, l, q_i, a_{ij}, \alpha_i$, *and*

$$f(m) \equiv l \pmod{k},$$
$$f(m) \not\equiv a_{ij} \pmod{q_i}, \qquad 1 \leqq i \leqq t,\ 1 \leqq j \leqq \alpha_i.$$

(c) *For* $0 \leqq t \leqq T$, $N_t(k) \geqq 0$.

(d) *There are* X, $C>0$, *independent of* k, *such that if we define* $F_t(k) = (X/k)\prod_{i=1}^{t}(1-\alpha_i/q_i)$, *then* $|N_0(k)-F_0(k)| < C$ *for all* k, l.

(e) $q_1 < q_2 < \cdots < q_T$.

(f) *We have chosen* t, Y *such that* $q_t \leqq Y$.

(g) *There is an* η, $0<\eta<1$, *such that for some* x_0, $10^3 \leqq x_0 \leqq e^{(\log Y)^\eta}$ *and*

$$\left| \sum_{q_i \leqq x} \frac{\alpha_i \log q_i}{q_i} - Q \log x \right| < \frac{A \log x}{(\log_2 x)^2}$$

for $x > x_0$.

(h) *For all* i, $0 < \alpha_i/q_i \leqq v < 1$, v *a constant.*

(i) $0.003eQ \log_2 Y \geqq 2$.

(j) *We have chosen* w *such that* $1 < w$, $eQ \log w \leqq 3(1-\eta)/2$.

(k) *We are using the abbreviations*

$$Z = \sum_{i=1}^{t} \frac{\alpha_i}{q_i}; \qquad W = \frac{36A(1-v) + 9AQ + 9A^2}{(1-v)\log_2 Y}; \qquad z = \frac{36A}{Q \log w \log_2 Y}.$$

(l) $Z \leqq 4Q(\log_2 Y)/3$.

(m) *We have chosen* n *an odd positive integer* $\geqq 2$.

Conclusion:

$$\begin{aligned} N_t(k) \leqq F_t(k)\left\{1 - (-1)^n \frac{e^{W+2z+2}}{(2\pi n)^{1/2} e^n (4 - w^Q(eQ\log w)^2)/4}\right\} \\ - (-1)^n \frac{X}{k} \frac{3Z}{4Q \log_2 Y} (4Qe \log_2 Y)/3 \\ - (-1)^n C Y^{n-1+2/(w-1)} e^Z. \end{aligned}$$

With the same hypothesis except that in (m), "odd" is replaced by "even," the conclusion holds with "$\leqq$" replaced by "$\geqq$".

In many applications of Brun's method the function g of (b) is defined by $g(m)=1$; in these cases $N_t(k)$ is simply the numbers of integers in a certain range having specified divisibility properties. The q's are frequently taken to be the successive primes.

2. **The order of** $\nu(n)$. The principal result of this section is contained in the following theorem:

THEOREM 1. *The number of positive integers* $m \leqq n$ *for which*

$$\nu(m) < \log_2 n + \omega(\log_2 n)^{1/2}$$

is $nD(\omega) + O(n \log_3 n/(\log_2 n)^{1/4})$.

As was pointed out in §1, this can be regarded either as a special case of the main theorem of [7] or as an extension of the principal result of [5]. The proof follows the lines of the latter paper; we shall preserve the notation used there, making the following definitions:

1. T denotes the closed interval $\{\log^6 n, n^{(\log_2 n)^{-3}}\}$,
2. $\nu'(m)$ the number of different prime factors of m which lie in T,
3. $q_1, q_2, \cdots, q_v$ symbols for the v primes q of T,
4. $a_1, a_2, \cdots$, the integers whose only prime divisors are q's,
5. $a_1^{(k)}, a_2^{(k)}, \cdots$, the integers whose factors are powers of k different q's ($k<2 \log_2 n$),
6. $A(m)$ the greatest a_i dividing m,
7. U_k the number of integers $m \leqq n$ for which $A(m)$ is an $a_i^{(k)}$,
8. $c_1, c_2, \cdots$ absolute constants,
9. $x=\sum_q 1/q$.

It is known [13, p. 102] that $\sum_{p\leqq y} 1/p=\log_2 y+C+O(1/\log y)$; since $x=\sum_{p<n^{(\log_2 n)^{-3}}} 1/p-\sum_{p<\log^6 n} 1/p$, it follows that

$$x = \log_2 n - 4 \log_3 n + O(1). \tag{1}$$

Lemma 1. *The number of integers $m \leqq n$ for which $\nu(m)-\nu'(m)>(\log_2 n)^{1/4}$ is $O(n \log_3 n/(\log_2 n)^{1/4})$.*

We have (see [9, p. 355]),

$$\begin{aligned}\sum_{m=1}^{n} (\nu(m) - \nu'(m)) &= \sum_{p\leqq n}\left[\frac{n}{p}\right] - \sum_{q}\left[\frac{n}{q}\right] \\ &= \sum_{p\leqq \log^6 n}\left[\frac{n}{p}\right] + \sum_{n^{(\log_2 n)^{-3}}<p\leqq n}\left[\frac{n}{p}\right] \\ &< n \sum_{p\leqq \log^6 n}\frac{1}{p} + n\sum_{p\leqq n}\frac{1}{p} \\ &\quad - n \sum_{p<n^{(\log_2 n)^{-3}}}\frac{1}{p} + n^{(\log_2 n)^{-3}} \\ &= n\{\log_3 n + \log_2 n + 3 \log_3 n - \log_2 n + O(1)\} \\ &= O(n \log_3 n),\end{aligned}$$

and similarly,

$$\begin{aligned}\sum_{m=1}^{n} (\nu(m) - \nu'(m)) &> n \sum_{p\leqq \log^6 n}\frac{1}{p} + n\sum_{p\leqq n}\frac{1}{p} - n \sum_{p\leqq n^{(\log_2 n)^{-3}}}\frac{1}{p} - \log^6 n - n \\ &= O(n \log_3 n).\end{aligned}$$

Hence the number of integers $m \leqq n$ such that

$$\nu(m) - \nu'(m) > (\log_2 n)^{1/4}$$

is

$$\frac{O(n \log_3 n)}{(\log_2 n)^{1/4}},$$

which gives the lemma.

LEMMA 2. *We have*

$$\frac{x^k}{k!} - \frac{3}{\log^5 n} < \sum_i{}' \frac{1}{a_i^{(k)}} < \frac{x^k}{k!},$$

where the dash on the summation means it is extended over the square-free $a_i^{(k)}$ *only.*

The proof is as in [5].

LEMMA 3. $U_k = ne^{-x}x^k/k! + O(n/\log^5 n)$.

The proof is as in [5].

LEMMA 4. *The number of integers* $m \leqq n$ *for which* $\nu'(m) < \log_2 n + \omega(\log_2 n)^{1/2}$ *is* $nD(\omega) + O(n \log_3 n/\log_2 n)$.

Clearly $\nu'(m) = \nu(A(m))$. Let us first consider the number of integers $m \leqq n$ for which $\nu'(m) < x + \omega x^{1/2}$; this is

$$\sum_{k < x + \omega x^{1/2}} U_k.$$

We put $y = x + \omega x^{1/2}$. By Lemma 3,

$$\sum_{k<y} U_k = ne^{-x} \sum_{k<y} \frac{x^k}{k!} + O\left(\frac{ny}{\log^5 n}\right).$$

It is known [1; 10] that there is a constant c_1 such that

$$\left| e^{-x} \sum_{k < x + \omega x^{1/2}} \frac{x^k}{k!} - D(\omega) \right| < \frac{c_1}{x^{1/2}},$$

and consequently

$$\begin{aligned} \sum_{k<y} U_k &= nD(\omega) + O(n/x^{1/2}) + O\left(\frac{ny}{\log^5 n}\right) \\ &= nD(\omega) + O(n/x^{1/2}) \end{aligned} \tag{2}$$

since $x \sim \log_2 n$.

We now consider the integers $m \leqq n$ for which

$$x + \omega x^{1/2} < \nu'(m) \leqq \log_2 n + \omega(\log_2 n)^{1/2}.$$

Since $x^k/k!$ assumes its maximum value for $k=[x]$, we see by Lemma 3 that the number of integers $m \leqq n$ for which $\nu'(m)=k$ is at most

$$ne^{-x}\frac{x^x}{x!}+O\left(\frac{n}{\log n}\right)<\frac{c_2 n}{x^{1/2}}. \tag{3}$$

Hence the number of $m \leqq n$ for which $x+\omega x^{1/2}<\nu'(m)\leqq \log_2 n+\omega(\log_2 n)^{1/2}$ is at most

$$\frac{c_2 n}{x^{1/2}}((\log_2 n - x)+\omega((\log_2 n)^{1/2}-x^{1/2})).$$

By (1), this is

$$O\left(\frac{n}{x^{1/2}}\left(\log_3 n+\omega\frac{\log_3 n}{(\log_2 n)^{1/2}+x^{1/2}}\right)\right)=O\left(\frac{n\log_3 n}{(\log_2 n)^{1/2}}\right).$$

This together with (2) completes the proof of the lemma.

We come now to the proof of the main theorem. By Lemma 4, we have only to prove that the number of $m\leqq n$ for which

$$\nu'(m)<\log_2 n+\omega(\log_2 n)^{1/2}$$

and

$$\nu(m)\geqq \log_2 n+\omega(\log_2 n)^{1/2}$$

is $O(n\log_3 n)/(\log_2 n)^{1/4})$.

We divide these integers into two classes; in the first class are those for which

$$\nu'(m)<\log_2 n+\omega(\log_2 n)^{1/2}-(\log_2 n)^{1/4},$$

of which there are $O(n\log_3 n/(\log_2 n)^{1/4})$ by Lemma 1, and those for which

$$\log_2 n+\omega(\log_2 n)^{1/2}-(\log_2 n)^{1/4}\leqq \nu'(m)<\log_2 n+\omega(\log_2 n)^{1/2},$$

and on account of (3) there are only

$$O\left((\log_2 n)^{1/4}\cdot\frac{n}{x^{1/2}}\right)=O\left(\frac{n}{(\log_2 n)^{1/4}}\right)$$

of these. This completes the proof of the theorem.

3. **Application to** $d(m)$. We now prove the following theorem.

THEOREM 2. *The number of integers* $m\leqq n$ *for which*

$$d(m)<2^{\log_2 n+\omega(\log_2 n)^{1/2}}$$

is also $nD(\omega)+O(n\log_3 n/(\log_2 n)^{1/4})$.

Let $k_n(\omega)$ be the number of $m\leqq n$ for which $\nu(m)<\log_2 n+\omega(\log_2 n)^{1/2}$,

$r_n(\omega)$ the number of $m \leqq n$ for which $d(m) < 2^{\log_2 n + \omega(\log_2 n)^{1/2}}$, and $p(n)$ the number of $m \leqq n$ for which $d(m)/2^{\nu(m)} < 2^{\epsilon(\log_2 n)^{1/2}}$. Then we have, as in [3],

$$k_n(\omega - \epsilon) - (n - p(n)) \leqq r_n(\omega) \leqq k_n(\omega),$$

for every $\epsilon > 0$. On account of Theorem 1, we have only to show that

$$k_n(\omega - \epsilon) = k_n(\omega) + O\left(\frac{n \log_3 n}{(\log_2 n)^{1/4}}\right)$$

and that

$$n - p(n) = O\left(\frac{n \log_3 n}{(\log_2 n)^{1/4}}\right)$$

for suitably chosen ϵ.

We define

$$\rho_k(m) = \begin{cases} 1 & \text{if } k \mid m \\ 0 & \text{if } k \nmid m. \end{cases}$$

Then clearly

$$d(m) = \prod_p (1 + \rho_p(m) + \rho_{p^2}(m) + \cdots),$$

$$2^{\nu(m)} = \prod_p (1 + \rho_p(m)).$$

Since

$$\frac{1}{1 + \rho_p(m)} = 1 - \frac{1}{2}\rho_p(m),$$

we have

$$\frac{d(m)}{2^{\nu(m)}} = \prod_p \left\{\left(1 - \frac{1}{2}\rho_p(m)\right)(1 + \rho_p(m) + \rho_{p^2}(m) + \cdots)\right\}$$

$$= \prod_p \left\{1 + \rho_p(m) + \rho_{p^2}(m) + \cdots - \rho_p(m) - \frac{1}{2}\rho_{p^2}(m) - \cdots\right\}$$

$$= \prod_p \left\{1 + \frac{1}{2}\rho_{p^2}(m) + \frac{1}{2}\rho_{p^3}(m) + \cdots\right\}.$$

Multiplying out, we get

$$\frac{d(m)}{2^{\nu(m)}} = 1 + \sum{}^* \frac{\rho_k(m)}{2^{\nu(k)}},$$

where the asterisk indicates that the summation is extended over all k which are such that if $p|k$ then $p^2|k$, for every p. Hence

$$\sum_{m=1}^{n} \frac{d(m)}{2^{\nu(m)}} = n + \sum{}^{*} \frac{1}{2^{\nu(k)}}\left[\frac{n}{k}\right] < n + n\sum{}^{*} \frac{1}{k\cdot 2^{\nu(k)}},$$

and we represent this convergent series by Σ^*, so that

$$\sum_{m=1}^{n} \frac{d(m)}{2^{\nu(m)}} < n(1 + \Sigma^*).$$

Now if there are μ integers m such that $0 < m \leqq n$ and

$$\frac{d(m)}{2^{\nu(m)}} > 2^{\epsilon(\log_2 n)^{1/2}},$$

then

$$\sum_{m=1}^{n} \frac{d(m)}{2^{\nu(m)}} > n - \mu + \mu\cdot 2^{\epsilon(\log_2 n)^{1/2}};$$

hence

$$n - \mu + \mu\cdot 2^{\epsilon(\log_2 n)^{1/2}} < n(1 + \Sigma^*),$$

from which we get

$$\mu < \frac{n\Sigma^*}{2^{\epsilon(\log_2 n)^{1/2}} - 1}\cdot$$

Consequently

$$p(n) = n - \mu > n\left(1 - \frac{\Sigma^*}{2^{\epsilon(\log_2 n)^{1/2}} - 1}\right),$$

and so

$$n - p(n) < \frac{n\Sigma^*}{2^{\epsilon(\log_2 n)^{1/2}} - 1}\cdot$$

We now take

$$\epsilon = \frac{\log_3 n}{\log 2 \log_2 n}$$

and get

$$n - p(n) < \frac{n\Sigma^*}{\log_2 n - 1} = O\left(\frac{n \log_3 n}{(\log_2 n)^{1/4}}\right).$$

Finally, it is clear that

$$k_n(\omega - \epsilon) = k_n(\omega) + O(\epsilon n) = k_n(\omega) + O\left(\frac{n \log_3 n}{(\log_2 n)^{1/4}}\right).$$

Thus we have proved Theorem 2.

4. **A general theorem.** We shall now prove the following theorem.

THEOREM 3. *Let $f(m)$ be a strongly additive function, that is, $f(mn)=f(m)+f(n)$ if $(m, n)=1$ and $f(p^n)=f(p)$. Suppose that $|f(p)|\leqq 1$ for all primes p. Let $\sum_p f^2(p)/p=\infty$, and let ω_1, ω_2 be any real numbers. Then the number of positive integers $m\leqq n$ for which simultaneously*

$$f(m) < A_n + \omega_1 B_n \quad \textit{and} \quad f(m+1) < A_n + \omega_2 B_n,$$

where $A_n=\sum_{p\leqq n} f(p)/p$, $B_n=(\sum_{p\leqq n} f^2(p)/p)^{1/2}$, is $n\cdot D(\omega_1)D(\omega_2)+o(n)$.

The argument used in the proof is a direct extension of that used in [7].

LEMMA 1. *Let $f_l(m)=\sum_{p|m,p<l} f(p)$. Then denoting by δ_l the density of positive integers for which simultaneously*

$$f_l(m) < A_l + \omega_1 B_l, \qquad f_l(m+1) < A_l + \omega_2 B_l,$$

one has

$$\lim_{l\to\infty} \delta_l = D(\omega_1)D(\omega_2).$$

Let

$$\rho_p(n) = \begin{cases} f(p) & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n. \end{cases}$$

Then clearly

$$f_l(m) = \sum_{p<l} \rho_p(m).$$

We divide the proof into two parts.

I. The functions $a\rho_p(m)+b\rho_p(m+1)$, where a, b are any fixed constants, not both zero, are statistically independent. To prove this, it suffices to show that

$$M\left\{e^{i\sum_{p\in P}(a\rho_p(m)+b\rho_p(m+1))}\right\} = \prod_{p\in P} M\left\{e^{i(a\rho_p(m)+b\rho_p(m+1))}\right\},$$

where $M\{t(m)\}=\lim_{n\to\infty}\sum_{m=1}^{n} t(m)/n$, and P is any set of primes all less than l. We give the proof only for the case where P consists of two primes p and q; the argument is no different in any essential respect in the general case. We prove then that

$$\begin{aligned} (4) \quad & M\left\{e^{i(a\rho_p(m)+b\rho_p(m+1)+a\rho_q(m)+b\rho_q(m+1))}\right\} \\ & = M\left\{e^{i(a\rho_p(m)+b\rho_p(m+1))}\right\}\cdot M\left\{e^{i(a\rho_q(m)+b\rho_q(m+1))}\right\}. \end{aligned}$$

Clearly,

$$e^{i a \rho_p(m)} = 1 + \frac{e^{ia\alpha} - 1}{\alpha} \rho_p(m),$$

where $\alpha = f(p)$, and similarly for $e^{ia\rho_q(m)}$. Hence the left side of (4) is

$$M\left\{\left(1 + \frac{e^{ia\alpha} - 1}{\alpha} \rho_p(m)\right)\left(1 + \frac{e^{ib\alpha} - 1}{\alpha} \rho_p(m+1)\right)\right.$$
$$\left.\cdot\left(1 + \frac{e^{ia\beta} - 1}{\beta} \rho_q(m)\right)\left(1 + \frac{e^{ib\beta} - 1}{\beta} \rho_q(m+1)\right)\right\}.$$

Let us put

$$\frac{e^{ia\alpha} - 1}{\alpha} = A_p, \quad \frac{e^{ib\alpha} - 1}{\alpha} = B_p, \quad \frac{e^{ia\beta} - 1}{\beta} = A_q, \quad \frac{e^{ib\beta} - 1}{\beta} = B_q;$$

then the product in the braces is, since $\rho_p(m)\rho_p(m+1) = 0$,

$$\begin{aligned}(1 + A_p\rho_p(m))&(1 + B_p\rho_p(m+1))(1 + A_q\rho_q(m))(1 + B_q\rho_q(m+1)) \\ &= (1 + A_p\rho_p(m) + B_p\rho_p(m+1))(1 + A_q\rho_q(m) + B_q\rho_q(m+1)) \\ &= 1 + A_p\rho_p(m) + B_p\rho_p(m+1) + A_q\rho_q(m) + B_q\rho_q(m+1) \\ &\quad + A_pA_q\rho_p(m)\rho_q(m) + B_pB_q\rho_p(m+1)\rho_q(m+1) \\ &\quad + A_pB_q\rho_p(m)\rho_q(m+1) + A_qB_p\rho_q(m)\rho_p(m+1),\end{aligned}$$

and the mean of this expression is

$$\begin{aligned}1 &+ \alpha A_p/p + \alpha B_p/p + \beta A_q/q + \beta B_q/q + (\alpha\beta/pq)(A_pA_q + B_pB_q) \\ &+ A_pB_q \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \rho_p(m)\rho_q(m+1) + A_qB_p \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \rho_p(m+1)\rho_q(m).\end{aligned}$$

We have

$$\rho_p(m)\rho_q(m+1) = \begin{cases}\alpha\beta & \text{if } m \equiv 0 \pmod{p} \text{ and } m \equiv -1 \pmod{q}, \\ 0 & \text{otherwise.}\end{cases}$$

The general solution of the pair of congruences $m \equiv 0 \pmod{p}$, $m \equiv -1 \pmod{q}$ is $m = m_0 + tpq$, where m_0 is the smallest positive solution, and there are $1 + [(n - m_0)/pq]$ solutions less than n. Hence

$$\rho_p(m)\rho_q(m+1) = \left(1 + \left[\frac{n - m_0}{pq}\right]\right)\alpha\beta,$$

and since $0 < m_0 < pq$,

$$\frac{\alpha\beta}{n}\left[\frac{n}{pq}\right] < \frac{1}{n} \sum_{m=1}^{n} \rho_p(m)\rho_q(m+1) < \frac{\alpha\beta}{n} + \frac{\alpha\beta}{n}\left[\frac{n}{pq}\right],$$

so that

$$A_pB_q \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \rho_p(m)\rho_q(m+1) = \frac{\alpha\beta A_pB_q}{pq}.$$

Similarly,

$$A_qB_p \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \rho_p(m+1)\rho_q(m) = \frac{\alpha\beta A_qB_p}{pq}.$$

Therefore, the mean of the left side of (4) is

$$(1 + \alpha A_p/p + \alpha B_p/p)(1 + \beta A_q/q + \beta B_q/q),$$

and this is obviously the same as the mean of the right side.

II. We can now prove that

$$M\left\{e^{i\xi(f_l(m)-A_l)/B_l+i\eta(f_l(m+1)-A_l)/B_l}\right\} \to e^{-(\xi^2+\eta^2)/2}$$

as $l\to\infty$. For clearly

$$\text{(5)}\qquad \begin{aligned} &M\left\{e^{i\xi(f_l(m)-A_l)/B_l+i\eta(f_l(m+1)-A_l)/B_l}\right\} \\ &\qquad\qquad = e^{(-iA_l/B_l)(\xi+\eta)}M\left\{e^{(i/B_l)(\xi f_l(m)+\eta f_l(m+1))}\right\}, \end{aligned}$$

$$\begin{aligned} e^{i(\xi f_l(m)/B_l+\eta f_l(m+1)/B_l)} &= e^{i(\xi\Sigma_{p\leqq l}\rho_p(m)/B_l+\eta\Sigma_{p\leqq l}\rho_p(m+1)/B_l)} \\ &= \prod_{p\leqq l} e^{i(\xi\rho_p(m)/B_l+\eta\rho_p(m+1)/B_l)}. \end{aligned}$$

Hence

$$M\left\{e^{i(\xi f_l(m)/B_l+\eta f_l(m+1)/B_l)}\right\} = M\left\{\prod_{p\leqq l} e^{i(\xi\rho_p(m)/B_l+\eta\rho_p(m+1)/B_l)}\right\},$$

and since by I the numbers $\xi\rho_p(m)/B_l+\eta\rho_p(m+1)/B_l$ $(p\leqq l)$ are independent, this is

$$\begin{aligned} &\prod_{p\leqq l} M\left\{e^{i(\xi\rho_p(m)/B_l+\eta\rho_p(m+1)/B_l)}\right\} \\ &\quad = \prod_{p\leqq l}\left(1 + \frac{e^{i\xi f(p)/B_l}-1}{p} + \frac{e^{i\eta f(p)/B_l}-1}{p}\right) \\ &\quad = \prod_{p\leqq l}\left(1 + \frac{if(p)}{pB_l}(\xi+\eta) - \frac{1}{2}\frac{f^2(p)}{pB_l^2}(\xi^2+\eta^2) - \frac{i}{6}\frac{f(p)}{pB_l^3}(\xi^3+\eta^3) + \cdots\right) \\ &\quad = \exp\left\{\sum_{p\leqq l}\log\left(1 + \frac{if(p)}{pB_l}(\xi+\eta) - \frac{1}{2}\frac{f^2(p)}{pB_l^2}(\xi^2+\eta^2)\right.\right. \\ &\qquad\qquad\qquad\qquad \left.\left. - \frac{i}{6}\frac{f^3(p)}{pB_l^3}(\xi^3+\eta^3) + \cdots\right)\right\}. \end{aligned}$$

By taking l sufficiently large, we can make

$$\left|\frac{if(p)}{pB_l}(\xi+\eta)-\frac{1}{2}\frac{f^2(p)}{pB_l^2}(\xi^2+\eta^2)-\frac{i}{6}\frac{f^3(p)}{pB_l^3}(\xi^3+\eta^3)+\cdots\right|<1,$$

since $B_l \to \infty$ with l. Then

$$\begin{aligned}\log\left(1+\frac{if(p)}{pB_l}(\xi+\eta)-\frac{1}{2}\frac{f^2(p)}{pB_l^2}(\xi^2+\eta^2)+\cdots\right)\\ =\frac{if(p)}{pB_l}(\xi+\eta)-\frac{1}{2}\frac{f^2(p)}{pB_l^2}(\xi^2+\eta^2)+\cdots\\ -\frac{1}{2}\left(\frac{if(p)}{pB_l}(\xi+\eta)-\frac{1}{2}\frac{f^2(p)}{pB_l^2}(\xi^2+\eta^2)+\cdots\right)^2+\cdots,\end{aligned}$$

so that

$$\begin{aligned}\sum_{p\leqq l}\log\left(1+\frac{if(p)}{pB_l}(\xi+\eta)+\cdots\right)\\ =\frac{i(\xi+\eta)}{B_l}\sum_{p\leqq l}\frac{f(p)}{p}-\frac{1}{2}\frac{(\xi^2+\eta^2)}{B_l^2}\sum_{p\leqq l}\frac{f^2(p)}{p}-\cdots\\ -\frac{1}{2}\sum_{p\leqq l}\frac{\left(\frac{if(p)}{B_l}(\xi+\eta)-\frac{1}{2}\frac{f^2(p)(\xi^2+\eta^2)}{B_l}+\cdots\right)^2}{p^2}+\cdots\\ =\frac{i(\xi+\eta)A_l}{B_l}-\frac{\xi^2+\eta^2}{2}-\frac{i(\xi^3+\eta^3)}{6}\frac{\sum_{p\leqq l}\frac{f^3(p)}{p}}{\sum_{p\leqq l}\left(\frac{f^2(p)}{p}\right)^{3/2}}+\cdots\\ -\frac{1}{2B_l^2}\sum_{p\leqq l}\frac{(if(p)(\xi+\eta)-\cdots)^2}{p^2}+\cdots,\end{aligned}$$

and all terms of this expression approach zero except the first two. Consequently, by (5),

$$\lim_{l\to\infty} M\{e^{(i/B_l)((f_l(m)-A_l)\xi+\eta(f_l(m+1)-A_l))}\}=e^{-(\xi^2+\eta^2)/2}.$$

Lemma 1 now follows immediately from the continuity theorem for Fourier-Stieltjes transforms (see [3, pp. 96–102]).

Now let $\Phi(n)$ be a positive function which tends to zero as $n\to\infty$ in such a way that

$$1/\Phi(n)=o((\log\log n)^2)$$

and also

$$1/\Phi(n) = o(B_n).$$

Let $n^{\Phi(n)} = \alpha_n$, $n^{(\Phi(n))^{1/2}} = \beta_n$; clearly $\alpha_n \to \infty$, $\beta_n \to \infty$ as $n \to \infty$. Let $a_1(n)$, $a_2(n), \cdots$, be the integers whose prime factors are all less than α_n, and let $\psi(m, n)$ be the greatest a_i which divides m. Let $q_1, q_2, \cdots, q_{T_n}$ be the primes less than α_n. We have the following lemma.

LEMMA 2. *The number of positive integers $m \leqq n$ for which*

$$\psi(m, n) = a_\kappa(n), \qquad \psi(m+1, n) = a_\lambda(n),$$

where $a_\kappa, a_\lambda \leqq \beta_n$, is zero if $(a_\kappa, a_\lambda) > 1$ or if $2 \nmid a_\kappa a_\lambda$ and otherwise is

$$\frac{n}{4\phi(a_\kappa a_\lambda)} \cdot \prod_{i=2}^{T_n} \left(1 - \frac{2}{q_i}\right) \prod_{2 \leqq i \leqq T_n, q_i | a_\kappa a_\lambda} \left(1 + \frac{1}{q_i(q_i - 2)}\right)(1 + o(1)),$$

where ϕ is Euler's ϕ-function.

This is clear if $(a_\kappa, a_\lambda) > 1$ or $2 \nmid a_\kappa a_\lambda$. Hence assume that $(a_\kappa, a_\lambda) = 1$, $2 \mid a_\kappa a_\lambda$. Then there is a unique integer r_0, $0 < r_0 < a$, such that

$$r_0 \cdot a_\kappa \equiv -1 \pmod{a_\lambda}. \tag{6}$$

Let

$$\frac{r_0 a_\kappa + 1}{a_\lambda} = b, \tag{7}$$

and consider the numbers r of the form

$$r = r_0 + ga \qquad (g = 0, 1, 2, \cdots). \tag{8}$$

$$\frac{r a_\kappa + 1}{a_\lambda} = b + g a_\kappa. \tag{9}$$

We wish to count the integers $m \leqq n$ for which

$$m = R a_\kappa, \qquad m + 1 = S a_\lambda,$$

where neither R nor S has any prime factors less than α_n. If this is the case, it must be that

$$R a_\kappa + 1 \equiv 0 \pmod{a_\lambda},$$

and R must be of the form of the r of (8). So we consider the numbers r and $(r a_\kappa + 1)/a_\lambda$, and ask that

$$r \not\equiv 0 \pmod{q_i}, \qquad \frac{r a_\kappa + 1}{a_\lambda} \not\equiv 0 \pmod{q_i} \qquad (i = 1, 2, \cdots, T_n).$$

From (8) we require then that

$$ga_\lambda \not\equiv -r_0 \pmod{q_i} \qquad (i = 1, 2, \cdots, T_n), \tag{10}$$

and from (9)

$$ga_\kappa \not\equiv -b \pmod{q_i} \qquad (i = 1, 2, \cdots, T_n). \tag{11}$$

If $q_i \mid a_\lambda$, (10) holds for every g, by (6); and if $q_i \mid a_\kappa$, (11) holds for every g, by (7). If $q_i \nmid a_\lambda$, (10) is equivalent to some restriction

$$g \not\equiv e_i \pmod{q_i},$$

and if $q_i \nmid a_\kappa$, (11) is equivalent to some restriction

$$g \not\equiv f_i \pmod{q_i}.$$

Moreover, $e_i \not\equiv f_i \pmod{q_i}$, for if the opposite were the case, we would have a g' such that

$$g'a_\lambda \equiv -r_0 \pmod{q_i}, \qquad g'a_\kappa \equiv -b \pmod{q_i},$$

and by (7),

$$g'a_\kappa \equiv -\frac{r_0 a_\kappa + 1}{a_\lambda},$$

$$g'a_\kappa a_\lambda \equiv -r_0 a_\kappa + 1,$$

$$-r_0 a_\kappa \equiv -r_0 a_\kappa + 1,$$

$$1 \equiv 0.$$

So we must count the integers g such that

$$0 < g \leqq \frac{n}{a_\kappa a_\lambda} - \frac{r_0}{a_\lambda} \qquad (\text{since } m = (r_0 + ga_\lambda)a_\kappa \leqq n), \tag{1}$$

$$g \equiv 0 \pmod 1, \tag{2}$$

$$\begin{aligned} &g \not\equiv e_i, f_i \pmod{q_i} && \text{if} \quad 1 \leqq i \leqq T_n,\ q_i \mid a_\kappa a_\lambda, \\ &g \not\equiv e_i \pmod{q_i} && \text{if} \quad 1 \leqq i \leqq T_n,\ q_i \mid a_\kappa,\ q_i \nmid a_\lambda, \\ &g \not\equiv f_i \pmod{q_i} && \text{if} \quad 1 \leqq i \leqq T_n,\ q_i \nmid a_\kappa,\ q_i \mid a_\lambda. \end{aligned} \tag{3}$$

We define $N_t(k)$, $0 \leqq t \leqq T_n$, to be the number of integers g which satisfy (1) and (3) (with T_n replaced by t) and

$$g \equiv l \pmod{k}, \tag{2'}$$

where $0 < l < k$, $(k, q_i) = 1$ for $i = 1, 2, \cdots, t$. Thus there are $N_{T_n}(1)$ integers of the kind specified in Lemma 2.

We also take, for $0 \leqq t \leqq T_n$,

$$F_t(k) = \frac{n}{ka_\kappa a_\lambda} \prod_{i=1}^{t} \left(1 - \frac{\alpha_i}{q_i}\right),$$

where

$$\alpha_i = \begin{cases} 1 & \text{if} \quad q_i \mid a_\kappa a_\lambda, \\ 2 & \text{otherwise;} \end{cases}$$

in particular, $\alpha_1 = 1$ (an empty product is unity, as usual). Then

$$N_0(k) = \left[\frac{n - a_\kappa r_0}{k a_\kappa a_\lambda}\right]$$

and

$$F_0(k) = \frac{n}{k a_\kappa a_\lambda},$$

so that

$$|N_0(k) - F_0(k)| < 2.$$

We now apply Theorem A; the following remarks and definitions apply to the corresponding hypotheses:

(a) We take $A = 1$, $Q = 2$.

(b) We take $T = \infty$, $f(m) = m$, $g(m) = 1$. (We have replaced m by q.) The fixed condition is

$$0 \leqq g \leqq X\left(= \frac{n}{a_\kappa a_\lambda}\right).$$

(d) $\quad C = 2.$

(f) $\quad t = \pi(\alpha_n) = T_n, \qquad Y = q_{T_n}.$

(g)
$$S = \sum_{q_i \leqq x} \frac{\alpha_i \log q_i}{q_i} - Q \log x < 2 \sum_{q_i \leqq x} \frac{\log q_i}{q_i} - 2 \log x$$
$$< 2\left(\frac{\log x}{x} + \frac{1}{x} + \frac{3}{2}\right) < 5.$$

Since the number of α's which are one is less than $\log X$,

$$S \geqq 2 \sum_{q_i \leqq x} \frac{\log q_i}{q_i} - \sum_{i=1}^{\log X} \frac{\log q_i}{q_i} - 2 \log x$$
$$> -2 \frac{\log x}{x} - 3 - \log_2 X - \frac{\log_2 X}{\log X} - \frac{1}{\log X} - \frac{3}{2}$$
$$> -\left(2 + 3 + \log_2 X + 1 + \frac{1}{2} + \frac{3}{2}\right) \geqq -(8 + \log_2 X),$$

so that

$$|S| < 8 + \log_2 X.$$

Take $\eta = 2/3$, $x_0 = e^{(\log_2 X)^2}$. Then $x_0 < e^{(\log Y)^\eta}$. Assume $\Phi(n) > (\log_2 X)^3/\log X$. Then if $x > x_0$, $X > X_1$ (that is, $n > n_1$),

$$8 + \log_2 X < \left(\frac{\log_2 X}{2 \log_3 X}\right)^2 = \frac{\log x_0}{(\log x_0)^2} < \frac{\log x}{(\log_2 x)^2},$$

and hence

$$|S| < \frac{\log x}{(\log_2 x)^2}$$

for $x > x_0$.

(h) Take $v = 2/3$.

(i) The condition $0.003\, eQ \log_2 Y > 2$ is certainly true for $X > X_2$ (that is, $n > n_2$).

(j) Take $1 < w < 6/5$.

(k) We have

$$Z \leqq \frac{5}{2} \log_2 \pi(x^\Phi), \qquad W \sim \frac{117}{\log_2 X}, \qquad z \sim \frac{18}{\log w \log_2 X^\Phi}.$$

(m) We shall replace n by ν; then $\nu \equiv 1 \pmod 2$, $\nu > 2$.

Since all hypotheses are satisfied, we infer the conclusion of Theorem A, for $n > \max(n_1, n_2)$. In order to obtain an asymptotic expression for $N_t(k)$, we must show that, for suitable ν, w, as $n \to \infty$,

I. $$\frac{e^{W+2z+2}}{(2\pi\nu)^{1/2} e^\nu (1 - w^Q e^2 Q^2 \log^2 w/4)} \to 0;$$

this implies that

$$\frac{e^{(\log w)-1}}{\nu^{1/2} e^\nu} \to 0.$$

II. $$\frac{X}{k}\left(\frac{Z}{4Q \log_2 Y/3}\right)^{4Qe \log_2 Y/3} = o(F_t(k));$$

this implies that

$$\frac{X}{(\Phi \log X)^{8/3}} = o(F_t(k)).$$

III. $$CY^{n-1+2/(w-1)} e^Z = o(F_t(k));$$

this implies that

$$q_t^{n-1+2/(w-1)} \log^{5/2} \pi(X^\Phi) = o(F_t(k)).$$

To satisfy these conditions, we choose

$$\nu = \begin{cases} [1/\Phi^{1/2}] & \text{if this is odd,} \\ 1 + [1/\Phi^{1/2}] & \text{otherwise} \end{cases}$$

and

$$w = 1 + \Phi^{1/2}.$$

I. We have

$$\frac{e^{(\log w)^{-1}}}{\nu^{1/2}e^{\nu}} = \frac{e^{(\log(1+\Phi^{1/2}))^{-1}}}{\Phi^{-1/2}e^{\Phi^{-1/2}}} < \frac{e^{(\Phi^{1/2}\Phi/2)^{-1}}}{e^{\Phi^{-1/2}}} \cdot \Phi^{1/2} = \Phi^{1/2}e^{1/2+\Phi^{1/2}/4+\cdots} \to 0.$$

II. Clearly

$$F_t(k) > \frac{X}{k}\prod_{i=1}^{t}\left(1 - \frac{2}{q_i}\right),$$

and it is known that for some constant $c>0$,

$$\prod_{i=1}^{t}\left(1 - \frac{2}{q_i}\right) \geqq \frac{c}{(\log q_t)^2}.$$

Hence

$$F_t(k) \geqq \frac{cX}{k\Phi^2 \log^2 X},$$

and therefore

$$\frac{X}{(\Phi \log x)^{8/3}} = o(F_t(k)).$$

III.
$$q_t^{\nu-1+2/(w-1)} \log^{5/2} \pi(X^{\Phi}) \leqq X^{\Phi(\Phi^{-1/2}-1+2\Phi^{-1/2})} \log^{5/2} X^{\Phi}$$
$$\leqq X^{3\Phi^{1/2}-\Phi} \log^{5/2} X = o(X^{\delta})$$

for every $\delta>0$. Hence this term is also $o(F_t(k))$.

Collecting these results, we infer from Theorem A that

$$N_t(1) < F_t(1) + o(F_t(1)),$$

and by making an obvious change in the definition of ν so as to make it even, we deduce that

$$N_t(1) > F_t(1) - o(F_t(1)),$$

so that finally

$$N_{T_n}(1) > F_{T_n}(1) - o(F_{T_n}(1)).$$

But

$$F_t(1) = X\prod_{i=1}^{t}\left(1-\frac{\alpha_i}{q_i}\right) = \frac{n}{a_\kappa a_\lambda}\prod_{i=1}^{t}\left(1-\frac{\alpha_i}{q_i}\right)$$

$$= \frac{n}{2a_\kappa a_\lambda}\prod_{i=2}^{t}\left(1-\frac{2}{q_i}\right)\cdot\frac{\prod_{2\leqq i\leqq t,\alpha_i=1}\left(1-\frac{1}{q_i}\right)}{\prod_{2\leqq i\leqq t,\alpha_i=1}\left(1-\frac{2}{q_i}\right)}$$

$$= \frac{n}{2a_\kappa a_\lambda}\prod_{i=2}^{t}\left(1-\frac{2}{q_i}\right)\prod_{2\leqq i\leqq t, q_i\mid a_\kappa a_\lambda}\left(1+\frac{1}{q_i-2}\right)$$

$$\cdot\frac{\prod_{2\leqq i\leqq t,q_i\mid a_\kappa a_\lambda}\left(1-\frac{1}{q_i}\right)}{\prod_{2\leqq i\leqq t,q_i\mid a_\kappa a_\lambda}\left(1-\frac{1}{q_i}\right)}$$

$$= \frac{n}{4\phi(a_\kappa a_\lambda)}\prod_{i=2}^{t}\left(1-\frac{2}{q_i}\right)\prod_{2\leqq i\leqq t,q_i\mid a_\kappa a_\lambda}\left(1+\frac{1}{q_i(q_i-2)}\right),$$

and putting $t=T_n$ we have the lemma.

LEMMA 3. *The number y of integers $\leqq M$ divisible by an $a_i(n)>\beta_n$ is less than $bM(\Phi(n))^{1/2}$, where b is an absolute constant. (It follows from this that the density of integers which are divisible by an $a_i(n)>\beta_n$ is less than $b(\Phi(n))^{1/2}$.)*

This is Lemma 4 of [7].

LEMMA 4. *Denote by l_n the number of positive integers $m\leqq n$ for which simultaneously*

$$f_{\alpha_n}(m) < A_{\alpha_n} + \omega_1 B_{\alpha_n}, \qquad f_{\alpha_n}(m+1) < A_{\alpha_n} + \omega_2 B_{\alpha_n}. \tag{12}$$

Then $l_n = nD(\omega_1)D(\omega_2)+o(n)$.

Divide the integers $m\leqq n$ which satisfy (12) into classes E_{11}, E_{12}, E_{21}, E_{13}, E_{22}, E_{31}, $\cdots$ so that $m\in E_{ij}$ if and only if simultaneously

$$\psi(m, n) = a_i(n), \qquad \psi(m+1, n) = a_j(n),$$

and denote by $|A|$ the number of elements of A. Clearly

$$l_n = \sum_{i,j}|E_{ij}| = \sum_{i,j;a_i,a_j\leqq\beta_n}|E_{ij}| + \sum_{i,j;a_i>\beta_n \text{ or } a_j>\beta_n}|E_{ij}|.$$

By Lemma 3,

$$\sum_{a_i>\beta_n \text{ or } a_j>\beta_n}|E_{ij}| < bn(\Phi(n))^{1/2},$$

and therefore it suffices to show that

$$\sum_{a_i, a_j \leqq \beta_n} |E_{ij}| = nD(\omega_1)D(\omega_2) + o(n)$$

as $n \to \infty$.

By Lemma 2,

$$(13)\qquad \begin{aligned} \sum_{a_i, a_j \leqq \beta_n} |E_{ij}| = \frac{n}{4} \prod_{i=2}^{T_n}\left(1 - \frac{2}{q_i}\right) \sum_{a_i, a_j \leqq \beta_n}{}' \frac{P(a_i, a_j, n)}{\phi(a_i a_j)} \\ + o\left(n \prod_{i=2}^{T_n}\left(1 - \frac{2}{q_i}\right) \sum_{a_i, a_j \leqq \beta_n}{}' \frac{P(a_i, a_j, n)}{\phi(a_i a_j)}\right), \end{aligned}$$

where the dash indicates a summation over the a_i, a_j satisfying

$$f_{\alpha_n}(a_i) < A_{\alpha_n} + \omega_1 B_{\alpha_n}, \qquad f_{\alpha_n}(a_j) < A_{\alpha_n} + \omega_2 B_{\alpha_n}, \qquad (a_i, a_j) = 1;\ 2 \mid a_i a_j,$$

$$P(a_i, a_j, n) = \prod_{2 \leqq i \leqq T_n, q_i \mid a_i a_j} \left(1 + \frac{1}{q_i(q_i - 2)}\right).$$

Now divide all the positive integers into classes F_{11}, $F_{12}, \cdots$ such that $m \in F_{ij}$ if and only if $\psi(m, n) = a_i(n)$, $\psi(m+1, n) = a_j(n)$, and let $\{F_{ij}\}$ be the density of F_{ij}. Now consider the set $\sum' F_{ij}$, with the dash as before. Putting $l = \alpha_n$ and using Lemma 1 we have that

$$\left\{\sum{}' F_{ij}\right\} = D(\omega_1)D(\omega_2) + o(1)$$

as $n \to \infty$.

Now

$$(14)\qquad \sum{}' F_{ij} = \sum_{a_i, a_j \leqq \beta_n}{}' F_{ij} + \sum_{a_i > \beta_n \text{ or } a_j > \beta_n}{}' F_{ij},$$

and by Lemma 3,

$$(15)\qquad \left\{\sum_{a_i > \beta_n \text{ or } a_j > \beta_n}{}' F_{ij}\right\} < b(\Phi(n))^{1/2}.$$

Furthermore, there are only a finite number of a's which are less than β_n, and hence

$$\left\{\sum_{a_i, a_j \leqq \beta_n}{}' F_{ij}\right\} = \sum_{a_i, a_j \leqq \beta_n}{}' \{F_{ij}\}.$$

But

$$\{F_{ij}\} = \begin{cases} 0 \quad \text{if} \quad (a_i, a_j) > 1 \text{ or } 2 \nmid a_i a_j, \\ \dfrac{n}{4\phi(a_i a_j)} \displaystyle\prod_{i=2}^{T_n}\left(1 - \frac{2}{q_i}\right) P(a_i, a_j, n), \end{cases}$$

and hence

$$\left\{\sum_{a_i,a_j\leqq\beta_n}{}' F_{ij}\right\} = \frac{n}{4}\prod_{i=2}^{T_n}\left(1-\frac{2}{q_i}\right)\sum_{a_i,a_j\leqq\beta_n}{}'\frac{P(a_i,a_j,n)}{\phi(a_ia_j)}. \tag{16}$$

Combining (14), (15), and (16) we have

$$\frac{n}{4}\prod_{i=2}^{T_n}\left(1-\frac{2}{q_i}\right)\sum_{a_i,a_j\leqq\beta_n}{}'\frac{P(a_i,a_j,n)}{\phi(a_ia_j)} = D(\omega_1)D(\omega_2)+o(1)$$

as $n\to\infty$. This and (13) give the lemma.

The remainder of the proof of Theorem 3 is almost identical with §5 of [7].

From Theorem 3 we get the following corollary.

COROLLARY. *The number of positive integers $m\leqq n$ for which*

$$\nu(m) < \log\log n + \omega_1(\log\log n)^{1/2},$$
$$\nu(m+1) < \log\log n + \omega_2(\log\log n)^{1/2}$$

is $nD(\omega_1)D(\omega_2)+o(n)$.

5. **Applications.** In this section we apply the corollary stated at the end of §4 to theorems concerning the relative sizes of $\nu(m)$, $\nu(m+1)$ and of $d(m)$, $d(m+1)$.

THEOREM 4. *Let $t_n(\omega)$ be the number of positive integers $m\leqq n$ for which $\nu(m)<\nu(m+1)+\omega(2\log\log n)^{1/2}$. Then $t_n(\omega)=nD(\omega)+o(n)$.*

Let us put $X_n(m)=\nu(m)-\log\log n/(\log\log n)^{1/2}$, $1\leqq m\leqq n$, then the corollary to Theorem 3 can be put in the form

$$\lim_{n\to\infty}\text{Prob}\left\{X_n(m)<\omega_1;\ X_n(m+1)<\omega_2\right\}=D(\omega_1)D(\omega_2).$$

Taking the characteristic functions, we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\exp\left(i\left(\xi\frac{\nu(m)-\log_2 n}{(\log_2 n)^{1/2}}+\eta\,\frac{\nu(m+1)-\log_2 n}{(\log_2 n)^{1/2}}\right)\right)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy.$$

Taking $\eta=-\xi$, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\exp\left(i\xi\,\frac{\nu(m)-\nu(m+1)}{(\log_2 n)^{1/2}}\right)\\ &\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\xi(x-y)\right)\exp\left(-\frac{x^2+y^2}{2}\right)dxdy.\end{aligned} \tag{17}$$

We put $t_n'(\omega) = t_n(\omega/2^{1/2})$; if

$$\frac{t_n'(\omega)}{n} \to \int_{-\infty}^{\infty} \rho(x)dx,$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \exp\left(i\xi \frac{\nu(m) - \nu(m+1)}{(\log_2 n)^{1/2}}\right) = \int_{-\infty}^{\infty} e^{i\xi x}\rho(x)dx. \tag{18}$$

Comparing (17) and (18), we see that we must write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\,(i\xi(x-y))\,\exp\left(-\frac{x^2+y^2}{2}\right)dxdy$$

in the form

$$\int_{-\infty}^{\infty} e^{i\xi x}\rho(x)dx.$$

Obviously

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\,(i\xi(x-y))\,\exp\left(-\frac{x^2+y^2}{2}\right)dxdy \\
&\quad = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \exp\left(i\xi x - \frac{x^2}{2}\right)dx\cdot\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \exp\left(-i\xi y - \frac{y^2}{2}\right)dy \\
&\quad = \left(\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \exp\left(i\xi x - \frac{x^2}{2}\right)dx\right)^2 \\
&\quad = \left(\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(x^2 - 2i\xi x - \xi^2) - \frac{\xi^2}{2}\right)dx\right)^2 \\
&\quad = \left(\frac{\exp\,(-\xi^2/2)}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \exp\,(-u^2/2)du\right)^2 \\
&\quad = \exp\,(-\xi^2).
\end{aligned}$$

Hence we must find a $\rho(x)$ such that

$$e^{-\xi^2} = \int_{-\infty}^{\infty} e^{i\xi x}\rho(x)dx.$$

Since

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{i\xi x}e^{-x^2/2}dx = e^{-\xi^2/2},$$

we put $\xi = 2^{1/2}\eta$, $x = y/2^{1/2}$, and get

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{i\eta y}\cdot\frac{e^{-y^2/4}}{2^{1/2}}\,dy = e^{-\xi}.$$

Hence $\rho(x)=1/2\pi^{1/2}e^{-x^2/4}$, and

$$t_n'(\omega)/n \to \frac{1}{2\pi^{1/2}}\int_{-\infty}^{\omega} e^{-x^2/4}dx,$$

which implies that $t_n(\omega)=nD(\omega)+o(n)$. This completes the proof.

THEOREM 5. *Let $r_n(\omega)$ denote the number of positive integers $m\leqq n$ for which*

$$d(m) < 2^{\omega(2\log_2 n)^{1/2}}d(m+1).$$

Then $r_n(\omega)=nD(\omega)+o(n)$.

We make the following definitions:

$$\begin{aligned}
f(m) &= \frac{d(m)2^{\nu(m+1)}}{d(m+1)2^{\nu(m)}},\\
g(n) &= 2^{\epsilon(2\log_2 n)^{1/2}} \qquad\qquad (\epsilon>0),\\
F_n &= \underset{m}{E}\left\{0<m\leqq n;\ \nu(m)<\nu(m+1)+(\omega-\epsilon)(2\log_2 n)^{1/2}\right\},\\
G_n &= \underset{m}{E}\left\{0<m\leqq n;\ f(m)\leqq g(n)\right\},\\
H_n &= \underset{m}{E}\left\{0<m\leqq n;\ d(m)<2^{\omega(2\log_2 n)^{1/2}}d(m+1)\right\},\\
p(n) &= |G_n|.
\end{aligned}$$

If m is in both F_n and G_n,

$$d(m) \leqq \frac{d(m+1)2^{\nu(m)}g(n)}{2^{\nu(m+1)}} \leqq d(m+1)2^{\omega(2\log_2 n)^{1/2}},$$

so that $m\in H_n$, and therefore $F_nG_n\subset H_n$. Since $|F_n|=t_n(\omega-\epsilon)$ and $|H_n|=r_n(\omega)$, it follows that

$$t_n(\omega-\epsilon)-(n-p(n)) \leqq r_n(\omega). \tag{19}$$

Since $d(m)/2^{\nu(m)}\geqq 1$, we have

$$\frac{1}{n}\sum_{m=1}^{n} f(m) \leqq \frac{1}{n}\sum_{m=1}^{n}\frac{d(m)}{2^{\nu(m)}},$$

so that (see [11])

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} f(m) < \infty.$$

Consequently $p(n)/n\to 1$ as $n\to\infty$, and from (19) we get

$$t_n(\omega - \epsilon) - o(n) = r_n(\omega). \tag{20}$$

Now let $F(m)$ be the number of (not necessarily distinct) prime divisors of m; it is easy to show that $2^{\nu(m)} \leqq d(m) \leqq 2^{F(m)}$. It is known (see [9, Theorem 435]) that there are constants B and C such that

$$\sum_{m=1}^{n} \nu(m) = n \log_2 n + Bn + o(n), \quad \sum_{m=1}^{n} F(m) = n \log_2 n + Cn + o(n).$$

Hence if $\theta(m) = F(m) - \nu(m)$,

$$\frac{1}{n} \sum_{m=1}^{n} \theta(m) = C - B + o(1),$$

so that $\theta(m)$ has finite mean.

We now put

$$\begin{aligned} s_n(\omega) &= \left| \underset{m}{E} \left\{ 0 < m \leqq n; \nu(m) < F(m+1) + \omega(2 \log_2 n)^{1/2} \right\} \right| \\ &= \left| \underset{m}{E} \left\{ 0 < m \leqq n; \nu(m) < \nu(m+1) \right. \right. \\ &\qquad \left. \left. + \left(\omega + \frac{\theta(m+1)}{(2 \log_2 n)^{1/2}} \right) (2 \log_2 n)^{1/2} \right\} \right|. \end{aligned}$$

Clearly

$$r_n(\omega) \leqq s_n(\omega). \tag{21}$$

Let

$$A_n = \underset{m}{E} \left\{ 0 < m \leqq n; \theta(m+1) > \log_3 n \right\};$$

since $\theta(m)$ has finite mean, $|A_n| = o(n)$. Moreover,

$$\begin{aligned} &\underset{m}{E} \left\{ 0 < m \leqq n; \nu(m) < \nu(m+1) + \left(\omega + \frac{\theta(m+1)}{(2 \log_2 n)^{1/2}} \right) (2 \log_2 n)^{1/2} \right\} \\ &\quad = \underset{m}{E} \left\{ 0 < m \leqq n; \nu(m) < \nu(m+1) \right. \\ &\qquad\qquad \left. + \left(\omega + \frac{\theta(m+1)}{(2 \log_2 n)^{1/2}} \right) (2 \log_2 n)^{1/2}; m \in A_n \right\} \\ &\qquad + \underset{m}{E} \left\{ 0 < m \leqq n; \nu(m) < \nu(m+1) \right. \\ &\qquad\qquad \left. + \left(\omega + \frac{\theta(m+1)}{(2 \log_2 n)^{1/2}} \right) (2 \log_2 n)^{1/2}; m \notin A_n \right\}. \end{aligned}$$

The number of elements in the first of these sets is $o(n)$, and the number of elements in the second set is $t_n(\omega+\epsilon')$, where

$$\epsilon' = \frac{\log_3 n}{(2 \log_2 n)^{1/2}} \to 0.$$

Hence

$$s_n(\omega) = t_n(\omega + \epsilon') + o(n),$$

and this with (20), (21) gives

$$t_n(\omega - \epsilon) - o(n) \leqq r_n(\omega) \leqq t_n(\omega + \epsilon') + o(n).$$

Since ϵ is arbitrary and $\epsilon' \to 0$ as $n \to \infty$, we conclude that $r_n(\omega) = nD(\omega) + o(n)$.

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